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Interreflection of light between parallel planes

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INTERREFLECTION OF LIGHT

BETWEEN

PARALLEL PLANES

by

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TABLE OF CONTENTS

PAGE

I. INTRODUCTION

The original plan for this thesis was to derive an analytical expression giving the illumination on any surface of a light court or well in terms of the dimensions and reflection factors of the various surfaces of the light court. Following this it was intended to verify the results by experimentation with a model of a light court. Preliminary study, however, showed it wiser to consider one-dimensional cases before proceeding to the more complicated one; hence it was decided to investigate other problems in interreflection involving parallel planes. Even the exact solutions of these cases are complicated by the necessity of integrating elliptic integrals of the third kind and of finding the solution of the Fredholm type of integral equation with infinite limits.

The general approach to the problem of interreflections, by the method of integral equations, is first discussed. The integral equations for several cases are established; namely, two parallel infinite half planes illuminated by a uniform diffuse plane source across one end, two parallel infinite planes illuminated by a lumihous rod parallel to the planes, two parallel infinite planes illuminated by a point source between them and finally those for the light court illuminated by a uniform diffuse sky or diffuse ceiling. All surfaces are considered to be matt i. e., the light which is emitted obeys the cosine law of emission.

While it is possible, by formal mathematics, to solve the integral equations in this thesis, they do not yield to easy solution. An approximation scheme is discussed and applied to one of the special cases.

The results obtained by approximation methods are compared with the experimentally obtained results of Meacock and Lambert.

II. REVIEW OF LITERATURE

The analytic treatment of interreflections was apparently begun in 1920 by S. P. Owen[12*], in his study of "On Radiation from a Cylindrical Wall". This work was followed in the same year by A. C. Bartlett's[1] simpler method. It remained for H. Buckley[2], however, to apply the method of integral equations to the problem of interreflection; he considered the interreflections in an infinitely long cylinder, a finite cylinder and a cylinder with a longitudinal slit. In later papers by Buckley[3,4] the interreflections in a finite cylinder illuminated by a uniform diffuse sky are discussed. The solution of all these cases by Buckley depends upon an exponential approximation of the kernel, a method devised by E. T. Whittaker[16]. The method of solution, however, is a modification of that suggested by Whittaker. In 1938 Whitmore[15] analyzed some of the above problems, using the differential equations resulting from the integral equations. Yamanauti[19] (often referred to as Yamauti) carried out the treatment of interreflections by a set of n linear equations and arrived at the Fredholm solution, although he did not explicitly state the integral equation. He has also studied the interreflections in an infinite cylinder[18].

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*All numbered references given in "Literature Cited".

All of the analytical work, so far, has been on interreflections in cylinders and spheres with two exceptions: (1) complicated graphical integration scheme devised by A. D. Moore[10] in 1926, to study the interreflections in a light court and (2) a particular case involving parallel planes considered by Buckley[3].

A subsequent paper by Manning and White[8] in 1930 treats another special case using Moore's method. As far as is known the present treatment is the first attempt to use integral equations in the case of parallel planes in which the luminosity of the planes is non-uniform.

In the field of experimental research Meacock and Lambert[9] in 1930 used a model to investigate the distribution of illumination in a light court. They developed an empirical equation for the distribution of illumination in terms of the dimensions of the light court, but this relationship is only directly applicable to points on the vertical center line of the walls of the light court. The results are, at best, approximate for all other points in the court.

All workers, whether in analytical or experimental research, have used matt surfaces, which lend themselves to ease of computation and in addition approach the actual conditions met in the practice of illuminating engineering.

III. INVESTIGATION

A. Analytical

1. Theory of interreflection.

Consider the surface shown in cross-section of figure 1. The point (x_1, y_1) of the surface remains fixed in space while the point (x, y) is variable. At each of these points there are incremental areas $d\sigma_1$ and $d\sigma$ with luminosities of $L(x_1, y_1)$ and $L(x, y)$. The subscript zero, e. g., $L_o(x,y)$ denotes the radiant flux per unit of area; this may be due to self-luminosity produced by transmission of flux, or to the temperature of the surface, but does not include the reflected flux.

In general there is a component of radiation due to interreflection. Each element of area $d\sigma$, having luminosity $L(x,y)$ produces an illumination at (x_1, y_1) equal to

$$L(x,y) \cdot K(x,y;x_1,y_1)\ d\sigma$$

where,

$K(x,y;x_1,y_1)$ = the illumination at (x_1, y_1) caused by unit luminosity of a unit area at (x, y).

K is called the kernel.

$L(x,y)$ = luminosity or radiant power emitted from a surface of unit area evaluated in terms of the standard visibility function.

The illumination at the point (x_1, y_1) due to the entire enclosure is

$$E(x_1, y_1) = \int_S L(x,y) \cdot K(x,y; x_1, y_1)\, d\sigma \quad \text{------------} \; 1$$

where the integration is taken over the surface visible from the point (x_1, y_1).

The resulting increase in the luminosity at (x_1, y_1) is given by equation 1 multiplied by the reflection factor of the surface at (x_1, y_1), i. e.,

$$L(x_1, y_1) = \rho(x_1, y_1) \int_S L(x,y) \cdot K(x,y; x_1, y_1)\, d\sigma$$

where, ρ = reflection factor (a numeric).

The total luminosity at (x_1, y_1) is

$$L(x_1, y_1) = L_o(x_1, y_1) + \rho(x_1, y_1) \int_S L(x,y) \cdot K(x, y; x_1, y_1)\, d\sigma \quad \text{----} \; 2$$

The total illumination at (x_1, y_1) is obtained from equation 2 by substituting

$$L = \rho E$$

Equation 2 becomes

$$E(x_1, y_1) = E_o(x_1, y_1) + \int_S \rho(x,y) \cdot E(x,y) \cdot K(x, y; x_1, y_1)\, d\sigma \quad \text{------} \; 3$$

where

$E(x,y)$ = the illumination at point (x, y) and is equal to the power received by each unit of area; the power is evaluated in terms of the standard visibility function.

Equations 2 and 3 are applicable in all cases, regardless of the configuration of the boundary surface.

2. <u>Integral equations</u>.

a. For two parallel infinite half planes illuminated by a uniform diffuse plane source across one end.

Figure 2 shows a cross-section of two parallel infinite half planes with the uniform diffuse plane source across one end. The planes are designated by A' and A" where the accents denote the upper and lower planes respectively.

The surfaces of the planes are assumed to be matt, i. e., perfectly diffusing. A perfectly diffusing surface may be defined as one which emits luminous flux according to the cosine law[11] (Moon p. 256)

$$I(\theta) = I_o \cos\theta \qquad \text{--- 4}$$

where

I_o = the candlepower of the surface (or any portion of the surface) normal to itself, ($\theta = o$)

$I(\theta)$ = the candlepower at an angle θ from the normal.

The direct component of illumination is determined in the following way. Consider figure 3 which shows a finite rectangular plane source. The width of the source is unity, while the length is expressed in terms of the angle subtended at the point being illuminated. The expression given by Moon[11](p. 324) for the illumination on a plane perpendicular to such a source and on a perpendicular through one corner of the source is

$$\frac{L}{2\pi}\left[\beta - \beta_1 \cos \gamma\right] \qquad (5)$$

where

L = luminosity of the uniform diffuse plane source

β, β_1 = angles subtended at the point by the length of the source as shown in figure 3.

γ = angle subtended at the point by the width of the source.

To obtain the illumination from an infinitely long plane source, β and β_1 are replaced by $\frac{\pi}{2}$ and the resulting expression is then multiplied by two. Expression 5 then becomes

$$\frac{L}{2}\left[1 - \sin \psi\right]$$

where

ψ = angle subtended at the source by the point being illuminated as is shown in figure 3.

The direct components of the illumination on the A' and A" of figure 2 are respectively

$$E'(x') = \frac{L}{2}\left[1 - \frac{x'}{\sqrt{1+x'^2}}\right] \text{ - - - - - - - - - - - - - - - - } 6$$

and

$$E''(x'') = \frac{L}{2}\left[1 - \frac{x''}{\sqrt{1+x''^2}}\right] \text{ - - - - - - - - - - - - - - - - - - } 7$$

where the accents indicate to which plane the quantities refer.

The component of illumination due to interreflection may be found from an expression given by Moon[11] (p.324). Figure 4 shows an incremental plane strip for which Moon gives the illumination, on a plane parallel to it, as

$$\frac{L\,d\xi}{2\pi r_o}\left[\cos\gamma \sin\gamma + \gamma\right] \sin^2\psi \text{ - - - - - - - - - - - - } 8$$

where

$d\xi$ = width of the incremental strip

r_o = distance from the source to the point being illuminated

γ = angle subtended, at the point illuminated, by the length of the source.

ψ = angle between the plane strip and r_o of figure 4.

The illumination from an infinitely long plane strip is obtained by doubling expression 8 and substituting $\frac{\pi}{2}$ for γ. For the A' and A" planes respectively, of figure 4, the illumination is

$$dE'(x') = \frac{L''(\xi'')}{2 r_o'} \cos^2\alpha' \, d\xi'' \text{ - - - - - - - - - - - - - - } 9$$

and

$$dE''(x'') = \frac{L'(\xi')}{2 r_o''} \cos^2 \alpha'' \, d\xi' \quad \text{------------} \quad 10$$

where $\alpha = 90° - \psi$

and the accents designate to which planes the quantities refer.

After replacing L by ρE, where ρ is the reflection factor, and expressing the angles and distances in terms of the coordinates x and y, equations 8 and 9 become

$$dE'(x') = \frac{\rho'' E''(\xi'')\, d\xi''}{2\left[1 + (\xi'' - x')^2\right]^{3/2}}$$

and

$$dE''(x'') = \frac{\rho' E'(\xi')\, d\xi'}{2\left[1 + (\xi' - x'')^2\right]^{3/2}}$$

On the A' plane the total illumination, due to interreflection, is

$$E'(x') = \frac{\rho''}{2} \int_0^\infty \frac{E''(\xi'') \cdot d\xi''}{\left[1 + (\xi'' - x')^2\right]^{3/2}} \quad \text{--------------} \quad 11$$

and on the A" plane it is

$$E''(x'') = \frac{\rho'}{2} \int_0^\infty \frac{E'(\xi')\, d\xi'}{\left[1 + (\xi' - x'')^2\right]^{3/2}} \quad \text{--------------} \quad 12$$

The total illumination is given by the sum of the direct component of illumination and the component of illumination due to interreflection.

On the A' plane the illumination is

$$E'(x') = \frac{L}{2}\left[1 - \frac{x'}{\sqrt{1+x'^2}}\right] + \frac{\rho''}{2}\int_0^\infty \frac{E''(\xi'')}{\left[1+(\xi''-x')^2\right]^{3/2}}\,d\xi'' \qquad 13$$

and on the A" plane the illumination is

$$E''(x'') = \frac{L}{2}\left[1 - \frac{x''}{\sqrt{1+x''^2}}\right] + \frac{\rho'}{2}\int_0^\infty \frac{E'(\xi')}{\left[1+(\xi'-x'')^2\right]^{3/2}}\,d\xi' \qquad 14$$

These two equations may be combined into one by substituting equation 13 and 14 (or vice versa).

If $\rho' = \rho'' = \rho$, equations 13 and 14 become

$$E(x) = \frac{L}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + \frac{\rho}{2}\int_0^\infty \frac{E(\xi)}{\left[1+(\xi-x)^2\right]^{3/2}}\,d\xi \qquad 15$$

An approximate solution of this equation is given in a later section because of its importance in illuminating engineering.

b. For two parallel infinite planes illuminated by a luminous rod parallel to the planes.

Consider the luminous rod shown in figure 5. Moon[11] (p. 324) gives the illumination from such a source in terms of the angle subtended by the length of the rod and the angle formed by the line joining the point illuminated and the plane through the axis of the rod parallel to the surface illuminated, as

$$\frac{L}{2\pi}\left[\cos\gamma \sin\gamma + \gamma\right] \cos\psi \qquad 16$$

where

L = luminosity of the rod

δ = diameter of the rod

r_0 = distance from center of rod to point illuminated

γ = angle subtended by rod at point illuminated.

ψ = angle formed by the line joining the point illuminated and the plane through the axis of the rod parallel to the surface illuminated.

After substitution of the appropriate angle $\frac{\pi}{2}$ and multiplication by two for an infinite rod, the above equation becomes, for the direct component of illumination on the A' and A" planes respectively, of figure 6

$$E'(x', t) = \frac{L\delta}{2r_0'} \cos\psi' = \frac{L \cdot \delta \cdot (1-t)}{2\left[x'^2 + (1-t)^2\right]} \quad \text{-------- } 17$$

and

$$E''(x'', t) = \frac{L\delta}{2r_0''} \cos\psi'' = \frac{L \cdot \delta \cdot t}{2\left[x''^2 + t^2\right]} \quad \text{-------- } 18$$

where

t = perpendicular distance from the A" plane to center of the source.

The accents indicate to which planes the quantities refer.

The component of illumination due to interreflection may be found from an expression given by Moon[11]. In this particular case it is identical with that given under section 2a for the case of two parallel infinite half planes illuminated by a uniform diffuse plane source, with the exception that the limits of integration in the present case are from minus infinity to plus infinity.

Hence the total illumination, on the A' plane from equations 11 and 17, is

$$E'(x',t) = \frac{L.\delta.(1-t)}{2\left[x'^2+(1-t)^2\right]} + \frac{\rho''}{2}\int_{-\infty}^{\infty}\frac{E''(\xi'',t)}{\left[1+(\xi''-x')^2\right]^{3/2}}\,d\xi'' \quad \text{------ 19}$$

and on the A" plane from equations 12 and 18, is

$$E''(x'',t) = \frac{L.\delta.t}{2\left[x''^2+t^2\right]} + \frac{\rho'}{2}\int_{-\infty}^{\infty}\frac{E'(\xi',t)}{\left[1+(\xi'-x'')^2\right]^{3/2}}\,d\xi' \quad \text{----- 20}$$

where, $E(x,t)$ designates the illumination at x for the source at t (see figure 6) and the accents indicate to which planes the quantities refer.

These two equations may be combined into one by substituting equation 19 in equation 20 (or vice versa).

If $\rho'=\rho''=\rho$ and $t=\frac{1}{2}$ the above equations become,

$$E(x,\tfrac{1}{2}) = \frac{L.\delta}{4\left[x^2+\frac{1^2}{4}\right]} + \frac{\rho}{2}\int_{-\infty}^{\infty}\frac{E(\xi,\frac{1}{2})}{\left[1+(\xi-x)^2\right]^{3/2}}\,d\xi \quad \text{------ 21}$$

c. For two parallel infinite planes illuminated by a point source between them.

Consider figure 7 which shows a cross-section of two parallel infinite planes illuminated by a point source which is placed a distance t up from the lower plane A". As in the previous cases the surfaces are assumed to be matt.

The direct component of illumination from the point source to a point on the upper plane A' and to a point on the lower plane A" are respectively

$$E'(x',t) = \frac{I \cdot (1-t)^2}{\left[(1-t)^2 + x'^2\right]^{3/2}} \qquad 22$$

and

$$E''(x'',t) = \frac{I \cdot t}{\left[t^2 + x''^2\right]^{3/2}} \qquad 23$$

where

I = intensity of the source; independent of direction.

and x = radius of the circle whose center is the foot of the perpendicular dropped from the source to the A' or A" plane.

The accents indicate to which planes the quantities refer.

The illumination due to interreflection may be obtained from an expression given by Moon[11]. Consider figure 8, in which the luminous source is a plane incremental circular band of radius ξ and width $d\xi$. The point at which the illumination is desired is one unit distant from the source along the perpendicular axis of the circular band and offset a distance x from the axis as shown. Moon[11] (p. 324) gives illumination at a point on a plane parallel to the source, as

$$dE(x) = \frac{2 \cdot L(\xi) \cdot \xi \;\; 1^2 (1 + x^2 + \xi^2)}{\left[(1 + x^2 + \xi^2)^2 - 4\xi^2 x^2\right]^{3/2}}$$

Applying this to the particular case at hand and substituting

$$L = \rho E$$

it is found that the illumination received from the A" by the A' plane is

$$E'(x') = 2\rho'' \int_0^\infty \frac{E''(\xi'') \cdot \xi'' \cdot (1+x'^2+\xi''^2)}{\left[(1+x'^2+\xi''^2)^2 - 4\xi''^2 x'^2\right]^{3/2}} \, d\xi'' \text{ --------- } 24$$

and from the A' by the A" plane is

$$E''(x'') = 2\rho' \int_0^\infty \frac{E'(\xi') \cdot \xi' \cdot (1+x''^2+\xi'^2)}{\left[(1+x''^2+\xi'^2)^2 - 4\xi'^2 x''^2\right]^{3/2}} \, d\xi' \text{ --------- } 25$$

where the accents indicate to which planes the quantities refer.

The total illumination at any point on the A' plane is

$$E'(x',t) = \frac{I \cdot (1-t)}{\left[x'^2 + (1-t)^2\right]^{3/2}} + 2\rho'' \int_0^\infty \frac{E''(\xi'',t) \cdot \xi'' \cdot (1+x'^2+\xi''^2)}{\left[(1+x'^2+\xi''^2)^2 - 4\xi''^2 x'^2\right]^{3/2}} \, d\xi'' \text{ --- } 26$$

and at any point on the A" plane is

$$E''(x'',t) = \frac{I \cdot t}{\left[x''^2 + t^2\right]^{3/2}} + 2\rho' \int_0^\infty \frac{E'(\xi') \cdot \xi' \cdot (1+x''^2+\xi'^2)}{\left[(1+x''^2+\xi'^2)^2 - 4\xi'^2 x''^2\right]^{3/2}} \, d\xi' \text{ --- } 27$$

These two equations may be combined into one by substituting equation 26 in equation 27 (or vice versa).

If $\rho' = \rho'' = \rho$ and $t = \frac{1}{2}$ the equation 26 and 27 become

$$E(x,\tfrac{1}{2}) = \frac{I}{2\left[x^2 + \frac{1}{4}\right]^{3/2}} + 2\rho \int_0^\infty \frac{E(\xi,\frac{1}{2}) \cdot \xi \cdot (1+x^2+\xi^2)}{\left[(1+x^2+\xi^2)^2 - 4\xi^2 x^2\right]^{3/2}} \, d\xi \text{ --- } 28$$

d. For a light court illuminated by a uniform diffuse sky or diffuse ceiling.

The problem of finding the illumination on any of the surfaces of a light court illuminated by a uniform sky, or a room illuminated by a uniform ceiling source, is a more difficult case; yet from the illuminating engineer's point of view most important.

The integral equations for this case, which are long and cumbersome, are not here explicitly stated in terms of the variables as was done in the preceding cases. The approach to the problem, however, is outlined.

Figure 9 shows the coordinate system used, the center of the ceiling being taken as the origin. The width of the court is 2b (y axis), the length 2c (z axis), and the depth a (x axis).

The point at which the illumination is desired is (x_0, y_0, z_0) and the variable point, on a boundary surface from which the incremental flux is emitted, is (x, y, z). As before the total illumination, given by the direct component plus the component of illumination due to interreflection, is

$$E(x_0, y_0, z_0) = \int_{\text{(ceiling)}} \frac{L(0, y, z) \cdot \cos\alpha \cdot \cos\beta}{\pi d^2} \, d\sigma$$

$$+ \int_{\text{(walls, floor, ceiling)}} \frac{\rho(x, y, z) \cdot E(x, y, z) \cos\varphi \cos\psi}{\pi r^2} \, d\sigma \qquad \text{------ 29}$$

where

d = distance from a point of the source to the point being illuminated.

r = distance from the surface reflecting light to the point being illuminated.

α = angle between the normal to the ceiling and the line joining a point of the ceiling to a point of the surface reflecting light.

β = angle between the normal to the surface reflecting light and the line joining a point of the ceiling and the point of the surface reflecting light.

φ = angle between the normal to the surface reflecting light and the line joining the point of the surface reflecting light and the point being illuminated.

ψ = angle between the normal to the surface of the point being illuminated and the line joining the point of the reflecting surface with the point being illuminated.

In the second integral the integration is carried out over all surfaces visible from the source and if the source is a reflecting surface (ceiling) the integration over this surface must also be included.

The first integral of equation 29 has been evaluated by Yamanauti[17]. However, since the results of the integration are not found in text books they are repeated here. The illumination at any point may be expressed in vector form. The magnitudes of the component vectors are given below; they represent the direct illumination.

The components in the x, y and z directions respectively, are

$$E'_x = \frac{L}{2\pi}\left[\frac{y_o - b}{\sqrt{x_o^2 + (y_o-b)^2}} \operatorname{Tan}^{-1} 2c \frac{\sqrt{x_o^2 + (y_o-b)^2}}{x_o^2 + z_o^2 + (y_o-b)^2 - c^2}\right.$$

$$+ \frac{y_o + b}{\sqrt{x_o^2 + (y_o+b)^2}} \operatorname{Tan}^{-1} 2c \frac{\sqrt{x_o^2 + (y_o+b)^2}}{x_o^2 + z_o^2 + (y_o+b)^2 - c^2}$$

$$+ \frac{z_o - c}{\sqrt{x_o^2 + (z_o-c)^2}} \operatorname{Tan}^{-1} 2b \frac{\sqrt{x_o^2 + (z_o-c)^2}}{x_o^2 + y_o^2 + (z_o-c)^2 - b^2}$$

$$\left. + \frac{z_o + c}{\sqrt{x_o^2 + (z_o+c)^2}} \operatorname{Tan}^{-1} 2b \frac{\sqrt{x_o^2 + (z_o+c)^2}}{x_o^2 + y_o^2 + (z_o+c)^2 - b^2}\right] \text{----} \quad 30$$

$$E'_y = \frac{L}{2\pi}\left[\frac{x_o}{\sqrt{x_o^2 + (y_o-b)^2}} \operatorname{Tan}^{-1} 2c \frac{\sqrt{x_o^2 + (y_o-b)^2}}{x_o^2 + z_o^2 + (y_o-b)^2 - c^2}\right.$$

$$\left. - \frac{x_o}{\sqrt{x_o^2 + (y_o+b)^2}} \operatorname{Tan}^{-1} 2c \frac{\sqrt{x_o^2 + (y_o+b)^2}}{x_o^2 + z_o^2 + (y_o+b)^2 - c^2}\right] \text{----} \quad 31$$

$$E'_z = \frac{L}{2\pi}\left[\frac{x_o}{\sqrt{x_o^2 + (z_o-c)^2}} \operatorname{Tan}^{-1} 2b \frac{\sqrt{x_o^2 + (z_o-c)^2}}{x_o^2 + y_o^2 + (z_o-c)^2 - b^2}\right.$$

$$\left. - \frac{x_o}{\sqrt{x_o^2 + (z_o+c)^2}} \operatorname{Tan}^{-1} 2b \frac{\sqrt{x_o^2 + (z_o+c)^2}}{x_o^2 + y_o^2 + (z_o+c)^2 - b^2}\right] \text{-----} \quad 32$$

where the accents indicate that the illumination is direct from the source to the point (x_0, y_0, z_0).

These equations in a somewhat modified form are given in Moon[11] (p. 264) in terms of the angles subtended by the point in question. They are also graphically represented by Moon[11] (p. 269) for ease of computation, but in this form, they are not applicable to the problem of interreflection.

In order to obtain the total illumination the following method of successive substitution was tried.

Consider the integral equation

$$\phi(x) = f(x) + \lambda \int_a^x K(x,s)\, \phi(s)\, ds \qquad \text{------------ } 33$$

where f(x), K(x) and λ are known.

To determine the solution of equation 33 let

$$\phi(x) = \sum_0^\infty \lambda^n \phi_n(x) \qquad \text{------------ } 34$$

The ϕ_n's are determined by placing the series in equation 33 and equating the coefficients of like powers of λ. They are

$$\phi_0(x) = f(x)$$

and

$$\phi_n(x) = \int_a^x K(x,s)\, \phi_{n-1}(s)\, ds \qquad \text{------------ } 35$$

If the series of equation 34 is uniformly convergent it is the solution of the integral equation 33.

A lengthy attempt was made to use this method of successive substitutions in an endeavor to solve the integral equations for the light court; but as five simultaneous integral equations are involved it has as yet not been possible to find even the second term of the respective series.

3. <u>Approximate solution of integral equations by exponential series.</u>

E. T. Whittaker[16], in his paper "On the Numerical Solution of Integral Equations" (1917), presents several methods for the solution of integral equations of the Abel and Poisson types. Since part of this work is applicable to the solution of two special cases discussed in this thesis, the method is reviewed here.

The Fredholm equation of the Poisson type

$$\phi(x) = f(x) + \lambda \int_a^b \phi(s) \, K(x-s) \, ds \qquad \text{--------- 36}$$

has the solution

$$\phi(x) = f(x) + \lambda \int_a^b \Gamma(x-s; \lambda) \cdot f(s) \cdot ds \qquad \text{-------- 37}$$

where the resolvent kernel satisfies the equation

$$\Gamma(x;\lambda) = K(x) + \lambda \int_a^b \Gamma(s;\lambda) \, K(x-s) \, ds \qquad \text{--------- 38}$$

For the particular type of equation which arises in the special case solved in this thesis Whittaker suggests the use of an exponential series as an approximation for the kernel. One great advantage in exponential approximation is that each exponential term involves two disposable constants, whereas a term of a polynomial involves only one disposable constant. It is therefore, Whittaker states, in general possible to obtain as high a degree of accuracy in approximation with <u>n</u> exponential terms as with a polynomial of <u>2n</u> terms. The existence of infinite limits also makes the polynomial objectionable.

While Whittaker outlines a method by Prony[13], and Buckley[3] refers the reader to the original work for means of determining the disposable constants it was found expedient to use a method devised by J. W. T. Walsh[14] and later simplified by Gheury de Bray[6]. The essentials of the method are given here for the case of a two term exponential sum.

Consider the curve in figure 10 which is to be analyzed into the exponential sum

$$y = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x} \qquad (39)$$

where the A's and α's are the constants to be determined.

Choosing four equidistant points a distance δ apart as shown in the figure; the corresponding equations involving x, where x is measured from the first ordinate chosen, are

$$y_0 = A_1 + A_2 \quad \text{------------------} \quad 40$$

$$y_1 = A_1 z_1 + A_2 z_2 \quad \text{------------------} \quad 41$$

$$y_2 = A_1 z_1^2 + A_2 z_2^2 \quad \text{------------------} \quad 42$$

$$y_3 = A_1 z_1^3 + A_2 z_2^3 \quad \text{------------------} \quad 43$$

where

$$z_i^x = e^{\alpha_i x} \quad \text{------------------} \quad 44$$

If the four equations 40, 41, 42 and 43 are solved for z it will be found that the values of z_1 and z_2 are the roots of the quadratic equation

$$z^2 + p_1 z + p_2 = 0$$

where the p's are constants. The z's may be determined after finding the p's, but this is unnecessary since Gheury de Bray has combined the steps; the result being the following equation.

$$(y_1^2 - y_0 y_2) z^2 + (y_3 y_0 - y_2 y_1) z + (y_2^2 - y_3 y_1) = 0 \quad \text{--------} \quad 45$$

from which the z's are determined directly.

Since $\qquad z_1^x = e^{\alpha_1 x}$

and $\qquad z_2^x = e^{\alpha_2 x}$

the α's may be determined by the use of logarithms.

To determine the A's the following are used

$$y_0 = A_1 + A_2 \quad \text{------------------} \quad 40$$

$$y_1 = A_1 z_1 + A_2 z_2 \quad \text{------------------} \quad 41$$

which, since the y's and z's are known, yield A_1 and A_2.

Gheury de Bray in addition to this work on the two term exponential sum includes a simple scheme for the determination of the coefficients in a three term exponential sum.

Let the kernel, the f(x) function and the resolvent kernel of equation 36, 37 and 38 be approximated by the following exponential expressions

$$K(x) = a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x} \qquad \text{---------- } 46$$

$$f(x) = c_1 e^{-\gamma_1 x} + c_2 e^{-\gamma_2 x} \qquad \text{---------- } 47$$

$$\Gamma(x;\lambda) = b_1 e^{-\beta_1 x} + b_2 e^{-\beta_2 x} \qquad \text{---------- } 48$$

where a's, b's, c's, α's, β's and γ's are the disposable constants. The knowledge of K and f makes the determination of these constants by some method of approximation possible. From the physical nature of the problem the α's and γ's will be positive.

Once the a's, c's, α's and γ's are found, the b's and β's are fixed by the following relationship.

Substitution of equations 46, 47 and 48 in equation 38 results in

$$b_1 e^{-\beta_1 x} + b_2 e^{-\beta_2 x} = a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x}$$
$$+ \lambda \int_0^\infty \left(b_1 e^{-\beta_1 s} + b_2 e^{-\beta_2 s}\right) \cdot \left(a_1 e^{-\alpha_1 |x-s|} + a_2 e^{-\alpha_2 |x-s|}\right) \cdot ds$$

$$= a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x}$$

$$+ \lambda \Bigg[e^{-\alpha_1 x} \left\{ \frac{a_1 b_1}{\beta_1 - \alpha_1} + \frac{a_1 b_2}{\beta_2 - \alpha_1} \right\} + e^{-\alpha_2 x} \left\{ \frac{a_2 b_1}{\beta_1 - \alpha_2} + \frac{a_2 b_2}{\beta_2 - \alpha_2} \right\}$$

$$+ e^{-\beta_1 x} \left\{ \frac{a_1 b_1}{\beta_1 + \alpha_1} + \frac{a_2 b_1}{\beta_1 + \alpha_2} - \frac{a_1 b_1}{\beta_1 - \alpha_1} - \frac{a_2 b_1}{\beta_1 - \alpha_2} \right\}$$

$$+ e^{-\beta_2 x} \left\{ \frac{a_1 b_2}{\beta_2 + \alpha_1} + \frac{a_2 b_2}{\beta_2 + \alpha_2} - \frac{a_1 b_2}{\beta_2 - \alpha_1} - \frac{a_2 b_2}{\beta_2 - \alpha_2} \right\} \Bigg]$$

By equating coefficients of like powers of e the following equations are obtained

$$\frac{2\alpha_1 a_1 \lambda}{\beta_1^2 - \alpha_1^2} + \frac{2\alpha_2 a_2 \lambda}{\beta_1^2 - \alpha_2^2} + 1 = 0 \qquad \text{------ 49}$$

$$\frac{2\alpha_1 a_1 \lambda}{\beta_2^2 - \alpha_1^2} + \frac{2\alpha_2 a_2 \lambda}{\beta_2^2 - \alpha_2^2} + 1 = 0 \qquad \text{------ 50}$$

$$\frac{\lambda}{\beta_1 - \alpha_1} b_1 + \frac{\lambda}{\beta_2 - \alpha_1} b_2 + 1 = 0 \qquad \text{------ 51}$$

$$\frac{\lambda}{\beta_1 - \alpha_2} b_1 + \frac{\lambda}{\beta_2 - \alpha_2} b_2 + 1 = 0 \qquad \text{------ 52}$$

The values of β_1 and β_2 are the square roots of the roots of

$$\frac{2\alpha_1 a_1 \lambda}{z - \alpha_1^2} + \frac{2\alpha_2 a_2 \lambda}{z - \alpha_2^2} + 1 = 0 \qquad \text{--- 53}$$

where z replaces β_1^2 and β_2^2 of equations 49 and 50.

In the particular example solved in the following section

$$\lambda = \rho/2$$

hence z for this particular case is

$$= \frac{\sqrt{[(\alpha_1 a_1 + \alpha_2 a_2)\rho - (\alpha_1^2 + \alpha_2^2)]^2 - 4[\alpha_1^2\alpha_2^2 - (\alpha_1\alpha_2^2 a_1 + \alpha_2\alpha_1^2 a_2)\rho]} - [(\alpha_1 a_1 + \alpha_2 a_2)\rho - \alpha_1^2 - \alpha_2^2]}{2} \qquad \text{--- 54}$$

and

$$\left.\begin{aligned} \beta_1 &= +\sqrt{z} \\ \beta_2 &= -\sqrt{z} \end{aligned}\right\} \qquad \text{--- 55}$$

From equations 51 and 52

$$b_1 = \frac{\alpha_1 - \alpha_2}{\frac{\rho}{2}\left[\frac{\beta_2 - \alpha_1}{\beta_1 - \alpha_1} - \frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_2}\right]} \qquad \text{--- 56}$$

and

$$b_2 = \frac{\alpha_1 - \alpha_2}{\frac{\rho}{2}\left[\frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_1} - \frac{\beta_1 - \alpha_2}{\beta_2 - \alpha_2}\right]} \qquad \text{--- 57}$$

Thus an approximation of the resolvent kernel is obtained. An approximate solution of the integral equation 36 is

$$\phi(x) = f(x) + \lambda \int_0^\infty \left\{ b_1 e^{-\beta_1|x-s|} + b_2 e^{-\beta_2|x-s|} \right\} \left\{ c_1 e^{-\gamma_1 x} + c_2 e^{-\gamma_2 x} \right\} ds$$

$$= f(x) + \lambda\left[e^{-\beta_1 x}\left\{\frac{b_1 c_1}{\gamma_1-\beta_1} + \frac{b_1 c_2}{\gamma_2-\beta_2}\right\} + e^{-\beta_2 x}\left\{\frac{b_2 c_1}{\gamma_1-\beta_2} + \frac{b_2 c_2}{\gamma_2-\beta_2}\right\}\right.$$

$$+ e^{-\gamma_1 x}\left\{\frac{b_1 c_1}{\gamma_1+\beta_1} + \frac{b_2 c_1}{\gamma_1+\beta_2} - \frac{b_1 c_1}{\gamma_1-\beta_1} - \frac{b_2 c_1}{\gamma_2-\beta_2}\right\}$$

$$\left.+ e^{-\gamma_2 x}\left\{\frac{b_1 c_2}{\gamma_2+\beta_1} + \frac{b_2 c_2}{\gamma_2+\beta_2} - \frac{b_1 c_2}{\gamma_2-\beta_1} - \frac{b_2 c_2}{\gamma_2-\beta_2}\right\}\right] \text{---} 58$$

where all the disposable constants are now known and may be substituted in the above equation.

4. <u>Approximate solution of 2a by exponential series</u>.

Equation 15 on page 14 expresses the illumination at any point on either plane of two parallel infinite half planes, illuminated by a uniform diffuse plane source, for the particular case in which the reflection factors of the planes are the same. The equation for this case is

$$E(x) = \frac{L}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + \frac{\rho}{2}\int_0^\infty \frac{E(\xi)}{\left[1+(\xi-x)^2\right]^{3/2}}\cdot d\xi \text{ ---- } 15$$

The solution of this equation would give the illumination in terms of the luminosity L. However, for practical purposes it is better to express the illumination at a point in terms of a unit luminosity of the source. Dividing both sides of the equation by L, which is assumed constant, the equation becomes

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + \frac{\rho}{2}\int_0^\infty \frac{\phi(\xi)}{\left[1 + (\xi - x)^2\right]^{3/2}} d\xi \text{ --------- 59}$$

where $\phi(x) = E(x)/L$ (a numeric) ------- 60

For the determination of the exponential sum, to approximate the direct component of illumination, $f(x)$ was set equal to $\frac{1}{2}F(x)$, i. e.,

$$f(x) = \frac{1}{2}F(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] \text{ ----------- 61}$$

where

$$F(x) = \left[1 - \frac{x}{\sqrt{1+x^2}}\right] \cong c_1 e^{-\gamma_1 x} + c_2 e^{-\gamma_2 x} \text{ ------ 62}$$

To determine the disposable constants c_1, c_2, γ_1 and γ_2 the method outlined on page 24 was used. Four equidistant points were chosen for values of x. The approximate and the exact values of $F(x)$, calculated from equation 62 for the particular values of x used, are given in Table I. The values of the y's shown were substituted in

$$(y_1^2 - y_0 y_2) z^2 + (y_3 y_0 - y_2 y_1) z + (y_2^2 - y_3 y_1) = 0 \text{ ------- 45}$$

for which the roots of z are

$$z_1 = 0.77862$$

$$z_2 = 0.252099$$

The disposable constants γ_1 and γ_2 were determined from

$$z^x = e^{\gamma_i x} \qquad (44) \qquad \text{------} \quad 63$$

which yields

$$\gamma_1 = -0.25040 \cong -0.2504$$
$$\gamma_2 = -1.377996 \cong -1.3780$$

The constants c_i were determined from

$$1 = A_1 + A_2 \qquad \text{------} \quad 40$$
$$y_1 = A_1 z_1 + A_2 z_2 \qquad \text{------} \quad 41$$

or

$$1 = c_1 + c_2$$
$$0.292893 = 0.778682\, c_1 + 0.252099\, c_2$$

which yields

$$c_1 \cong 0.0775$$
$$c_2 \cong 0.9225$$

Hence the two term exponential approximation

$$F(x) \cong 0.0775\, e^{-0.2504x} + 0.9225\, e^{-1.37800x} \qquad \text{------} \quad 64$$

Values of F(x) calculated by slide rule using this approximation are given in Table I under the heading marked $F(x) \cong$ Graph 2 shows the curves for the exact and the approximate functions for F(x).

It may be remarked that this approximation enters into the solution of the integral equation only in the determination of the resolvent kernel. For the final expression of the approximate solution; see equations 68 to 71.

To determine the disposable constants in a two term exponential sum for the approximation of the kernel the same procedure as above was used.

The kernel

$$K(x) = \frac{1}{[1+x^2]^{3/2}} \qquad \text{------------} \qquad 65$$

was approximated by:

$$K(x) \cong a_1 e^{-\alpha_1 x} + a_2 e^{-\alpha_2 x}$$
$$\cong 1.5500\, e^{-1.4720x} - 0.5500\, e^{-5.9955x} \qquad \text{------} \qquad 66$$

In Table II are shown the four equidistant points for values of x which were chosen. In addition are shown values of the exact expression for the kernel together with the results obtained from the exponential approximation. The latter are given under the heading $K \cong$ and were computed by slide rule. Graph 1 shows the exact and approximation curves for K(x).

Since all the disposable constants of equation 54 have been determined (with the exception of ρ , which may be given any value) the β's can be obtained for each value of ρ . This was done for ρ equal to 0.2, 0.4, 0.6 and 0.8. The b's were determined from equations 56 and 57.

Hence the solution of

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + \frac{\rho}{2}\int_0^\infty \frac{\phi(\xi)}{[1+(\xi-x)^2]^{3/2}}\, d\xi \qquad \text{-----} \qquad 59$$

is

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right]$$

$$+ \frac{\rho}{4}\left[e^{-\beta_1 x}\left\{\frac{b_1 c_1}{\gamma_1 - \beta_1} + \frac{b_1 c_2}{\gamma_2 - \beta_1}\right\} + e^{-\beta_2 x}\left\{\frac{b_2 c_1}{\gamma_1 - \beta_2} + \frac{b_2 c_2}{\gamma_2 - \beta_2}\right\}\right.$$

$$+ e^{-\gamma_1 x}\left\{\frac{b_1 c_1}{\gamma_1 + \beta_1} + \frac{b_2 c_1}{\gamma_1 + \beta_2} - \frac{b_1 c_1}{\gamma_1 - \beta_1} - \frac{b_2 c_1}{\gamma_1 - \beta_2}\right\}$$

$$\left. + e^{-\gamma_2 x}\left\{\frac{b_1 c_2}{\gamma_2 + \beta_1} + \frac{b_2 c_2}{\gamma_2 + \beta_2} - \frac{b_1 c_2}{\gamma_2 - \beta_1} - \frac{b_2 c_2}{\gamma_2 - \beta_2}\right\}\right] \quad \text{--- } 67$$

Equation 67 becomes for

$\rho = 0.2$

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + 0.0055\, e^{-6.4903x} + 1.0913\, e^{-1.3111x}$$
$$+ 0.0091\, e^{-0.2504x} - 1.0782\, e^{-1.3780x} \quad \text{----} 68$$

$\rho = 0.4$

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + 0.0103\, e^{-6.1013x} + 0.6435\, e^{-1.1347x}$$
$$+ 0.0224\, e^{-0.2504x} - 0.6131\, e^{-1.3780x} \quad \text{---} 69$$

$\rho = 0.6$

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + 0.0154\, e^{-6.1519x} + 0.5123\, e^{-0.9332x}$$
$$+ 0.0449\, e^{-0.2504x} - 0.0461\, e^{-1.3780x} \quad \text{---} 70$$

$\rho = 0.8$

$$\phi(x) = \frac{1}{2}\left[1 - \frac{x}{\sqrt{1+x^2}}\right] + 0.0180\, e^{-6.2006x} + 0.4371\, e^{-0.6863x} + 0.0963\, e^{-0.2504x} - 0.3630\, e^{-1.3780x} \quad \text{---71}$$

The calculated values of the ϕ's for the various reflection factors are given in Table III. They are also presented in the form of curves in graphs 3 and 4.

B. Experimental

The experimental work of H. F. Meacock and G. E. V. Lambert.

In 1930, Meacock and Lambert[9] published the results of their investigation of "The Efficiency of Light Wells" (courts). Their work is wholly experimental and thus provides a means for verification of the results obtained by the analytical methods used in this thesis.

In their investigation they used a model light court in which court sizes varying from one to ten units in well length and one-half to ten units in well depth could be obtained; a width of unity for the well was held fixed. In all cases the surfaces were painted with various shades of grey matt paint with reflection factors varying from 24.6 to 80 percent.

The data are given in several different ways in the form of graphs to show the effect of each variable on the illumination in the light court.

A light court with opening 1 x 10 and depth of 10 units, from the information given by Meacock and Lambert, approaches the condition that would be obtained by two parallel infinite half planes; hence the experimental results of this case form a basis for verification of the exponential approximations used in this thesis.

For ease of comparison the experimental and analytical results for $\rho = 0.8$ have been plotted on graph 3, the full line curves are from analysis while the circular points are from the experimental work of Meacock and Lambert. Also shown on graph 3 are the analytical results for ρ equal to 0, 0.2, 0.4 and 0.6.

In addition to the graphical presentation of their results, Meacock and Lambert, have established the following empirical formula for the interreflection component of illumination

$$E_R = \left(0.4541 \times 10^{-yd}\right) \cdot \rho^m \quad \text{-------------} \quad 72$$

where

$y = 0.117 + (0.149/l)$

ρ = reflection factor

$m = 1.519 + 0.33d$

$l = \dfrac{L}{B}$

$d = \dfrac{x}{B}$

L = length of well opening

B = width of well opening

X = depth of observation on the center line of the largest wall of the well.

Meacock and Lambert state that equation 72 represents the observed results to an accuracy of about 10 percent over the whole range of dimensions and reflection factors investigated, except when the daylight factor ($\phi(x)$) is below 1 percent, such a condition, after all, is of slight practical importance.

IV. DISCUSSION

The close agreement of the experimental and analytical results is clearly shown on graph 3 and in Table IV. As the use of a great number of curves would have complicated graph 3 and thereby detracted from its usefulness, only the experimental results for $\rho = 0.8$ are indicated. The agreement is so close that the experimental results are given just as points (indicated by small circles), no curve being drawn through them; the full line curve is the analytical result for $\rho = 0.8$. Table IV gives a comparison between the experimental and analytical results for ρ equal to 0.635, 0.428 and 0.246 the values used by Meacock and Lambert. The experimental data are taken directly from the graphs given by them, while the analytical results are obtained from graph 4, although the curves for x equal to 1.5, 2.5 and 3.5 (the values used by Meacock and Lambert) are not shown. The agreement is very satisfactory, especially in view of errors unavoidably introduced in their experiments, in plotting, and in reading graphs.

The only verification of the analytical method used in this thesis is experimental. An estimation of the error in the approximations for K and f, beyond that obtainable by inspection from graphs 1 and 2 and Tables I and II, was considered. However, since there are no analytical means for determining the maximum error in the solution of an integral equation resulting from an approximation of K and f, it is felt that rigid determination of the errors in these functions is unnecessary.

The most common scheme for obtaining an approximation of a known function is to use polynomials. It has however, been pointed out that each term of an exponential series has two disposable constants, whereas each term of a polynomial has only one disposable constant. Hence, the same accuracy may be expected, in general, from an *n* term exponential series as from a 2*n* term polynomial. It was found, in the particular case solved in this thesis, that the exponential series met the boundary conditions satisfactorily and the integrations were simplified. Buckley in his work on interreflections has used exponential approximations extensively, with gratifying results.

In his doctoral thesis, "Use of Functionals in Obtaining Approximate Solutions of Linear Operational Equations" G. L. Gross discusses the general method of solution of all types of linear equations by approximation. The generality of his method is so broad that all explicit approximation schemes are apparently but special cases. The work of E. T. Whittaker and Prescott Crout[5] are examples.

While Whittaker and Crout give definite directions as to how to proceed in the solution of certain types of equations, neither give any definite suggestions as to means of determining the error created by the approximations.

By modification, the method used in determining the solution of equation 15 can be extended to solve equation 28, which is the equation for two parallel infinite planes illuminated by a luminous rod parallel to the planes. The kernel is the same as that for the problem solved; the only differences between the two equations are in the functions and the limits of integration. For equation 16 the limits of integration are from minus infinity to plus infinity.

It seems possible that the method of exponential approximation might be used in the solution of the problem of the light court. There is a great deal of similarity between it and the problem of the two parallel half-planes. The source in the light court, however, is finite and hence the angles β_1 and β_2 of equation 5 cannot be replaced by $\pi/2$. As a first approximation the illumination along the vertical center line of each wall could be assumed equal to that at each point of the wall having the same depth. The method of successive substitutions as outlined on page 22 could then be used to determine the successive components of illumination due to interreflection.

V. LITERATURE CITED

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VI. ACKNOWLEDGMENT

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VII. DIAGRAMS

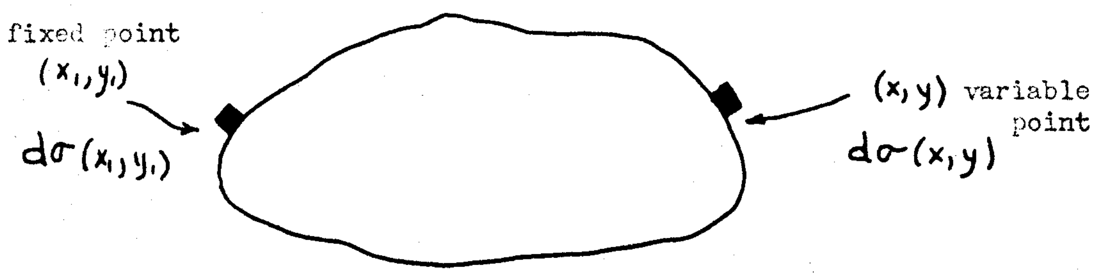


Figure 1 Cross section of luminous surface.

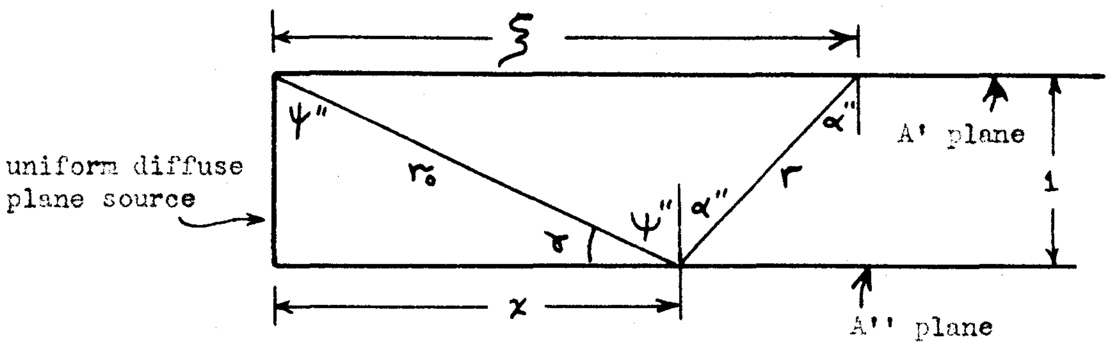


Figure 2 Cross section of two infinite half-planes illuminated by source across one end.

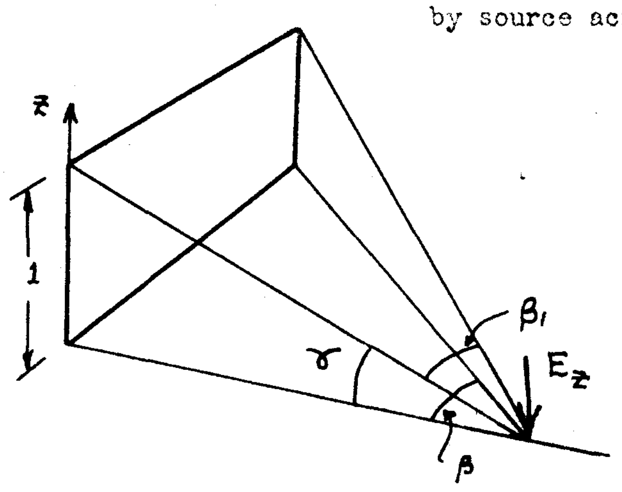


Figure 3 Direct illumination from uniform diffuse plane source.

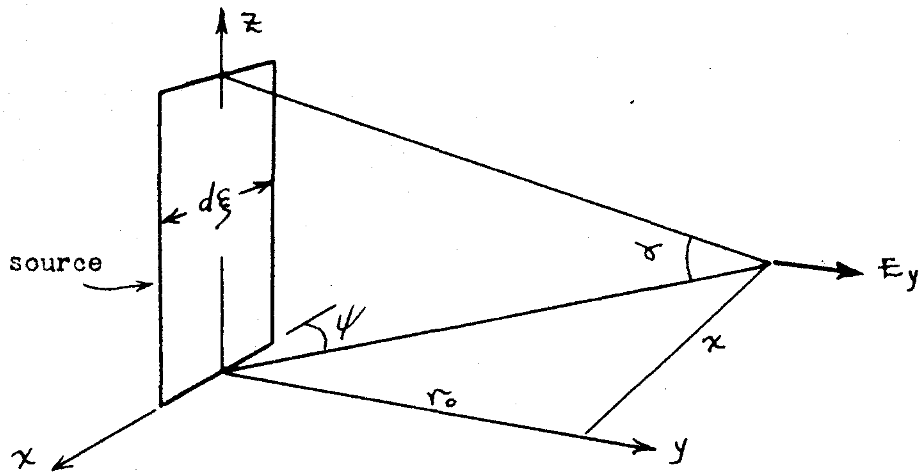


Figure 4 Illumination from an incremental strip.

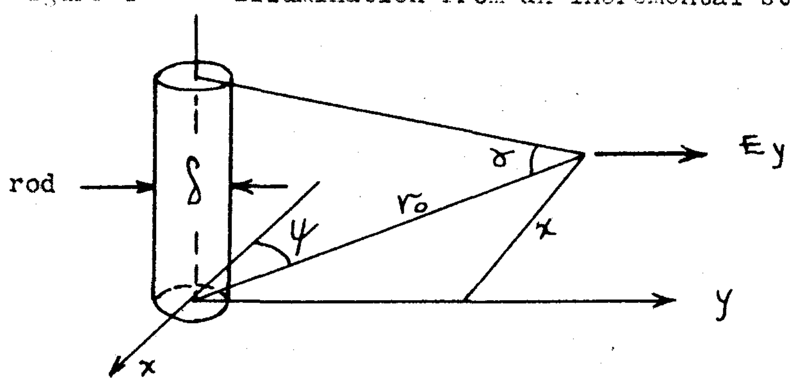


Figure 5 Illumination from a luminous rod.

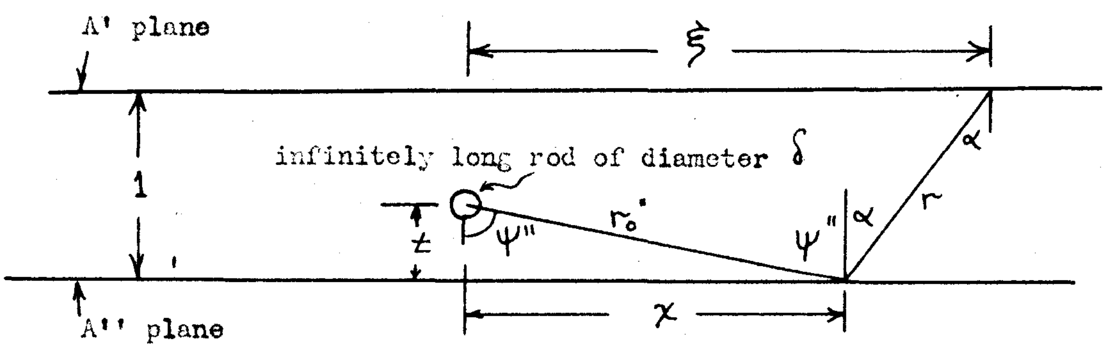


Figure 6 Cross section of two parallel infinite planes illuminated by luminous rod.

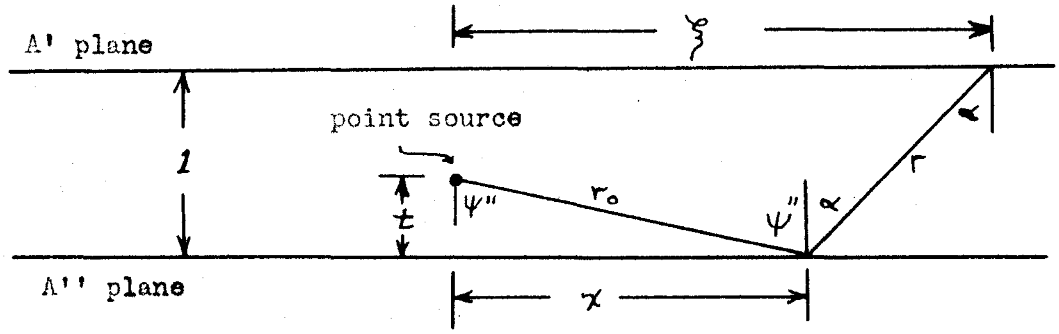


Figure 7 Cross section of two parallel infinite planes illuminated by point source.

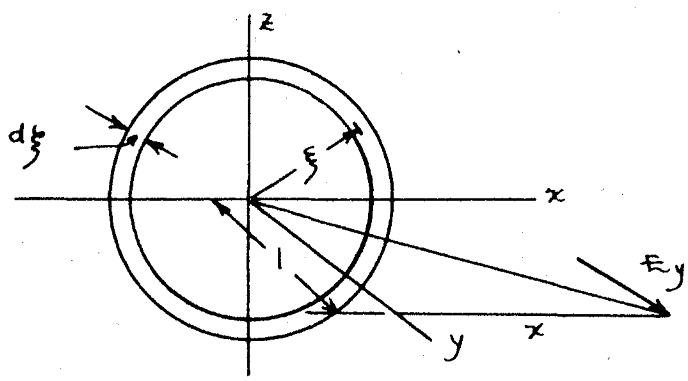


Figure 8 Illumination from a plane circular band.

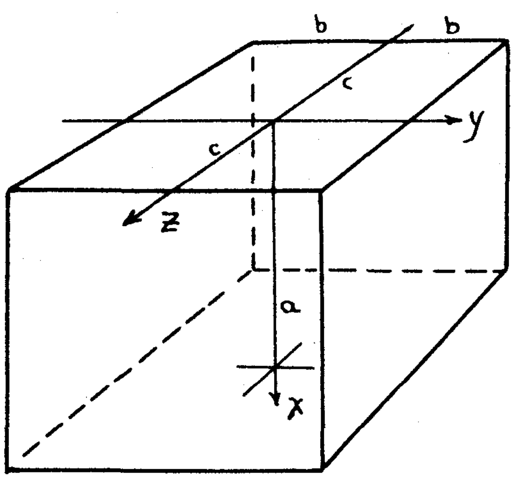


Figure 9 Coordinate system of light court.

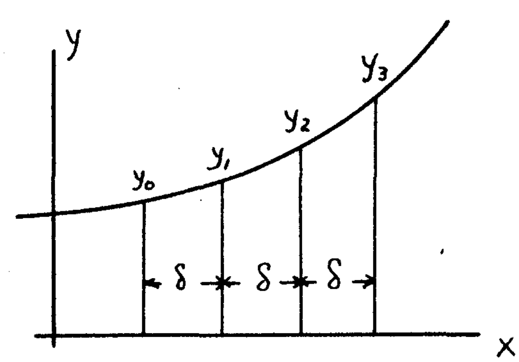


Figure 10 Curve to be analyzed.

TABLE I

Exact and approximate values of F(x)

x	F(x)	F(x) ≅*
0	$y_0 = 1$	1
0.5	0.552786	0.531
1	$y_1 = 0.292893$	0.293
2	$y_2 = 0.105573$	0.105
3	$y_3 = 0.051317$	0.0515
4	0.029857	0.0322
6	0.013606	0.0198

* $F(x) \cong 0.0775e^{-0.250x} + 0.9225e^{-1.378x}$

The approximate values given were calculated by slide rule.

TABLE II

Exact and approximate values of K(x)

x	K(x)	K(x) ≅*
0	y_0 = 1	1
0.25	0.909348	0.948
0.5	y_1 = 0.715541	0.715
1	y_2 = 0.353553	0.355
1.5	y_3 = 0.170674	0.172
2	0.089443	0.0819
3	0.0316227	0.0188
6	0.004443	0.0002

* $K(x) \cong 1.55e^{-1.4720x} - 0.55e^{-5.996x}$
The approximate values given were calculated by slide rule.

TABLE III

The calculated values of $\phi(x)$ for various ρ's

x	$\rho=0$	$\rho=0.2$	$\rho=0.4$	$\rho=0.6$	$\rho=0.8$
0	0.5	0.5272	0.5630	0.6120	0.6884
0.5	0.2764	0.3079	0.3529	0.4057	0.4908
1	0.1465	0.1768	0.2164	0.2678	0.3506
2	0.0528	0.0694	0.0941	0.1298	0.1993
3	0.0257	0.0340	0.0479	0.0707	0.1211
4	0.0149	0.0196	0.0275	0.0419	0.0767
6	0.0068	0.0090	0.0124	0.0186	0.0353

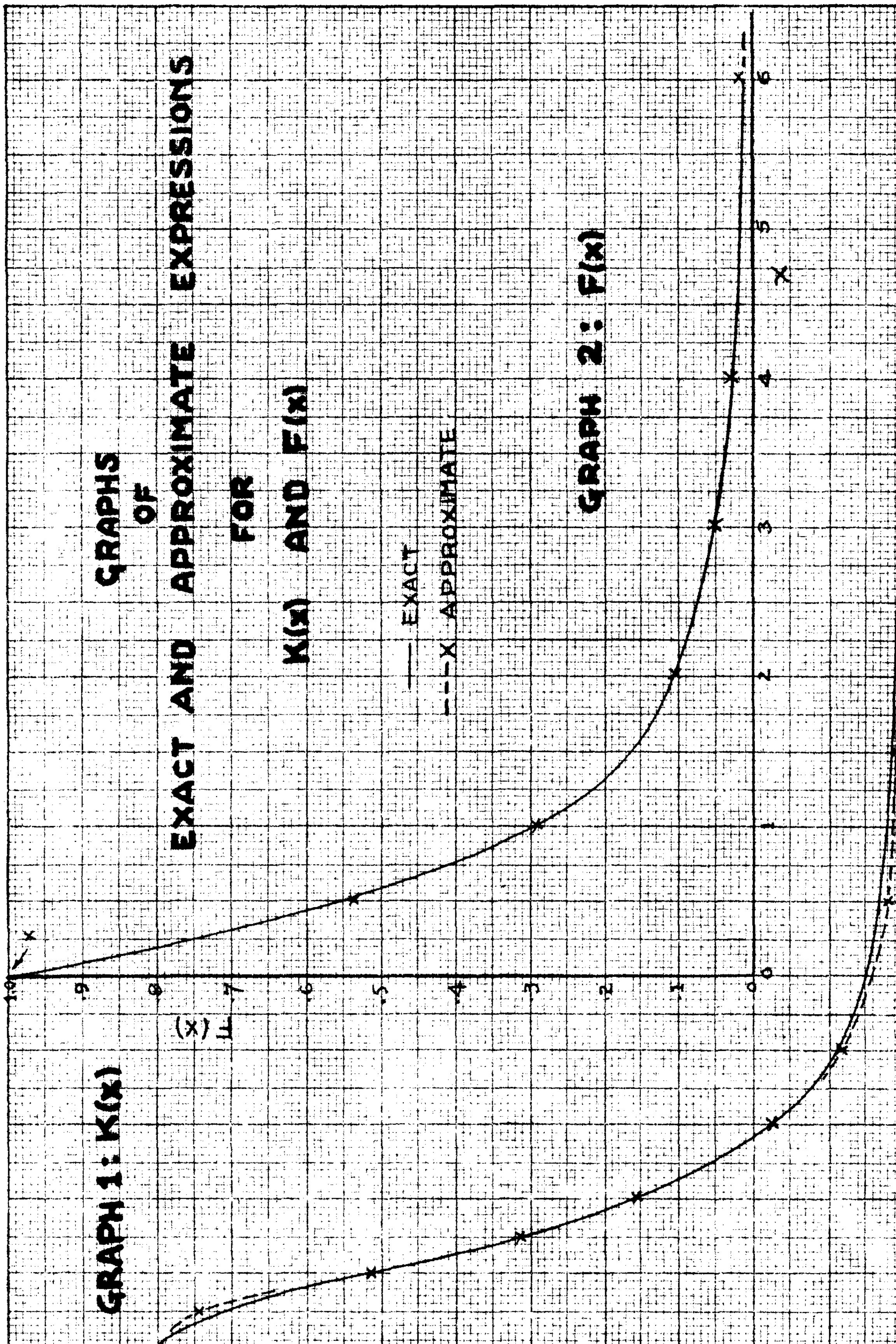
TABLE IV

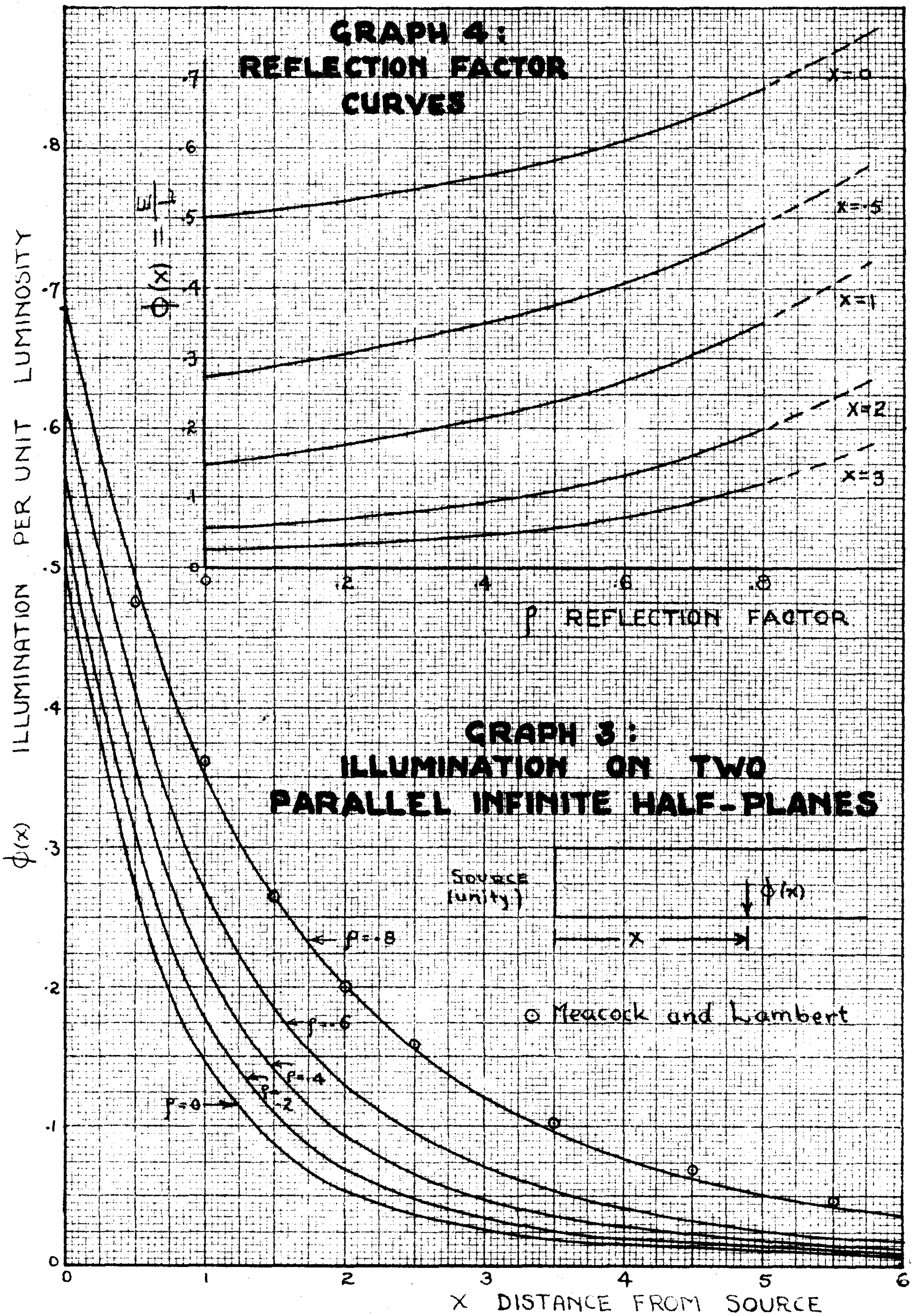
Comparison between analytical and experimental results

	Reflection factor							
	80 %		63.5 %		42.8 %		24.6 %	
x	Experimental[1]	Analytical[2]	Experimental	Analytical	Experimental	Analytical	Experimental	Analytical
0.5	0.475	0.49	0.44	0.42	0.34	0.355	0.292	0.315
1	0.362	0.35	0.295	0.28	0.22	0.22	0.171	0.184
1.5	0.265	0.265	0.205	0.199	0.14	0.147	0.105	0.125
2.5	0.16	0.156	0.105	0.105	0.058	0.067	0.042	0.05
3.5	0.103	0.096	0.06	0.06	0.028	0.038	0.021	0.027
4.5	0.069	0.063						
5.5	0.048	0.043						

[1] Experimental results of Meacock and Lambert[9].

[2] Analytical results obtained from graphs 3 and 4 of this thesis.


GRAPHS
OF
EXACT AND APPROXIMATE EXPRESSIONS
FOR
K(x) AND F(x)
— EXACT
---X APPROXIMATE
GRAPH 1: K(x)
GRAPH 2: F(x)
K(X)
F(X)
X


GRAPH 4:
REFLECTION FACTOR
CURVES
$\phi(x) = \frac{1}{E}$
X=0
X=.5
X=1
X=2
X=3
ρ REFLECTION FACTOR
GRAPH 3:
ILLUMINATION ON TWO
PARALLEL INFINITE HALF-PLANES
Source (unity)
φ(x)
X
ρ=.8
ρ=.6
ρ=.4
ρ=.2
ρ=0
○ Meacock and Lambert
φ(x) ILLUMINATION PER UNIT LUMINOSITY
X DISTANCE FROM SOURCE